Lecture Notes on Simple Interest, Compound Interest, and Future Values

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Outcomes

• Understanding what is meant by "the time value of money".

• Understanding the relation between present and future values.

• Calculating the simple and compound interests and the corresponding future and present values of an amount of money invested today.
Outline

• Time value of money and interest
• The simple interest; Present and Future values
• The compound interest; Present and Future values
• Compounding more than once a year
Time Value of Money and Interest

• Which would you prefer $10,000 today or $10,000 in 5 years? Obviously, $10,000 today; you already recognize that there is time value of money.

• If you put some money on a bank account for a year, then the bank can do whatever it wants with that money for a year; RIGHT? To reward you for that, the bank pays you some interest. The more years the bank works with your money, the more rewards you would expect to get as an accumulated interest!
Why Time

Why is time such an important element in your decision?

Time allows you the opportunity to postpone consumption and earn compensation for lending your money as an interest.
Types of Interest

• Simple Interest ($SI$); which is interest earned on only the original amount, called Principal, lent over a period of time at a certain rate.

• Compound Interest ($CI$); which is interest earned on any previous interests earned as well as on the Principal lent.
Simple Interest Formula

The Simple interest is given by

\[ SI = P_0 \cdot i \cdot n, \]

where

\( P_0 \): Present value today (deposited \( t = 0 \)),
\( i \): interest rate per period of time,
\( n \): number of time periods
Simple Interest Example

Assume that you deposit $1000 in an account paying 7% annual simple interest for 2 years. What is the accumulated interest at the end of the second year?

Solution: \[ SI = P_0 \cdot i \cdot n = 1000 \cdot \frac{7}{100} \cdot 2 = $140. \]
Simple Interest and Future Value

Future Value (FV) is the value at some future time of a present amount of money evaluated at a given interest rate.

What is the future value of the deposit?

\[ FV_n = P_0 + SI = P_0 (1 + n \cdot i). \]

For our example above, \( FV_n = 1000 + 140 = \$1140. \)

Note that there are 4 variables in the formula above. Therefore, having any three of them, could be used to find the fourth one.
Simple Interest and Present Value

Present Value (PV) is the current value of a future amount of money evaluated at a given interest rate. What is the present value of the previous problem? The present value is simply the $1000 you originally deposited. That is the value today.
Why Compound Interest

![Graph comparing future values with different interest rates over time. The graph shows the growth of an investment with 10% compound interest, 7% compound interest, 10% simple interest, and 7% simple interest over various years.]
Future Value Formula

Assume you deposit $1000 at an annual compound interest rate of 7% for 2 years.

First Year: \[ FV_1 = P_0(1 + 1 \cdot i) = P_0(1 + i)^1 = 1000(1.07) = $1070 \]
You earned $70 interest on your $1000 deposit over the first year.
This is the same amount of interest you would earn under the simple interest.

Second Year: \[ FV_2 = P_0(1 + i)(1 + i) = P_0(1 + i)^2 = 1000(1.07)^2 = $1144.90 \]
You earned extra $4.90 in year two with compound interest over simple interest.
General Future Value Formula

\[ FV_1 = P_0(1 + i)^1 \]
\[ FV_2 = P_0(1 + i)^2 \]
\[ \vdots \]
\[ FV_n = P_0(1 + i)^n \]

where again,

- \( P_0 \): Present value today (deposited \( t = 0 \)),
- \( i \): interest rate per period of time,
- \( n \): number of compounding periods.

As we have seen in the Figure, the Future value here is growing exponentially.

Note also that we have again 4 variables as for \( SI \).
Example

A person wants to know how large his deposit of $10000 today will become at a compound annual interest rate of 10\% for 5 years.

Solution: Using the formula:

\[ FV_5 = 10000(1 + 0.1)^5 = $16105.10. \]
Double Your Money

How long does it take to double $5000 at a compound rate of 12% per year?

Solution: Using the formula, $FV_n = P_0(1 + i)^n$:
$10000 = 5000(1 + 0.12)^n$, i.e., $2P_0 = P_0(1 + 0.12)^n$.

Hence,

$$2 = (1.12)^n \iff \ln(n) = n \ln(1.12)$$

$$\iff n = \frac{\ln(2)}{\ln(1.12)}$$

$$\iff n = 6.1163 \approx 6.12,$$

where the natural logarithmic function is used to solve the equation;

$$\ln \equiv \log.$$
It does not matter how much money you have at the start.

Therefore, \[ n = \frac{\ln(2)}{\ln(i+1)} \approx \frac{\ln(2)}{i}, \] for small \( i \), where we use the first term of the Taylor expansion of \( \ln(i + 1) \) about \( i = 0 \) as an approximation for \( \ln(i + 1) \) for small \( i \).
Frequency of Compounding

General Formula:
\[ FV_n = P_0(1 + i)^n \equiv P_0(1 + \frac{r}{m})^{mt}, \]
where
- \( FV_n \): Future Value,
- \( P_0 \): Principal, Present value today (deposited \( t = 0 \)),
- \( r \): annual interest rate,
- \( m \): number of compounding periods per year.
- \( t \): time; in years,
- \( i \): interest rate per period of time,
- \( n \): total number of compounding periods.

Note that \( i := \frac{r}{m} \), and \( n = mt \), so, if interest is compounded annually, then \( i = r \) and \( n = t \).
Impact of Frequency

A person has $1000 to invest for 2 years at an annual compound interest rate of 12%:

- **annual**: \( FV_2 = 1000 \left(1 + \frac{0.12}{1}\right)^{(1)(2)} = $1254.40 \)
- **semi-annual**: \( FV_2 = 1000 \left(1 + \frac{0.12}{2}\right)^{(2)(2)} = $1262.48 \)
- **quarterly**: \( FV_2 = 1000 \left(1 + \frac{0.12}{4}\right)^{(4)(2)} = $1266.77 \)
- **monthly**: \( FV_2 = 1000 \left(1 + \frac{0.12}{12}\right)^{(12)(2)} = $1269.73 \)
- **daily**: \( FV_2 = 1000 \left(1 + \frac{0.12}{365}\right)^{(365)(2)} = $1271.20 \)
Present Value

Assume that you need $1000 in 2 years. Let us find how much you need to deposit today at a rate of 7% compounded annually.

\[ FV_n = P_0(1 + i)^n, \] which implies, \[ PV = P_0 = \frac{FV_n}{(1+i)^n}. \]

Thus, \[ PV = \frac{1000}{(1.07)^2} = 873.4387 \cdots \approx$873.44. \]

As a general Present Value Formula:

\[ PV = \frac{FV_n}{(1+i)^n}. \]
Example

A person wants to know how large of a deposit to make so that the money will grow to $10000 in 5 years at a rate of 10% compounded annually.

solution:

\[ PV = \frac{10000}{(1+0.1)^5} = 6209.21323 \cdots \approx $6209.21. \]
Comparing \(SI\) and \(CI\)

- Suppose that you put your money \(m\) years in one account and then \(n\) years in another account, and that both accounts pay

  Compound interest at a rate \(i\).

  Simple Interest at rate \(i\).

For \(CI\) in \(m\) years, we have \(FV_m = P_0(1 + i)^m\), and then you withdraw the money and put it in another account for \(n\) years and get

\[
FV_n = P_0(1 + i)^m(1 + i)^n = P_0(1 + i)^{m+n}.
\]

This is the same as what you would get if you had kept the Principal in the same account for \(m + n\) years.
Now, for SI in \((b)\), we have
\[ FV_m = P_0 + IS_m = P_0 + P_0 \cdot i \cdot m. \]
If you withdraw the money and put it in another account for \(n\) years you get
\[
FV_n = (P_0 + P_0 \cdot i \cdot m) + SI_n \\
= (P_0 + P_0 \cdot i \cdot m) + (P_0 + P_0 \cdot i \cdot m) \cdot i \cdot n \\
= P_0 + P_0 \cdot i \cdot m + P_0 \cdot i \cdot n + P_0 \cdot i^2 \cdot m \cdot n.
\]
Whereas, you would get if you kept the Principal in the same account for \(m + n\) years
\[
FV_{m+n} = P_0 + P_0 \cdot i \cdot (m+n) = P_0 + P_0 \cdot i \cdot m + P_0 \cdot i \cdot n.
\]
You could increase the interest you earn by withdrawing your money halfway and open a new account with the same simple interest rate $i$. This inconsistency means that simple interest is not that often used in practice.
- Simple interest: Principal after $n$ years grows LINEARLY; $P_0(1 + i \cdot n)$.
- Compound interest: Principal after $n$ years grows EXPONENTIALLY; $P_0(1 + i)^n$.
- There is no difference between Simple interest and Compound interest in 1 year; both lead to $P_0(1 + i)$. 