SOME CLASSES OF STAR COVERING SPACES

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DEDICATION

This work is dedicated to whom I loved my mother, my father, my sisters and my brothers. I dedicate this work also to my supervisor Prof. Dr. Adnan Al-Bsoul.

I dedicate my thesis to whom read this!

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ABSTRACT

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In this thesis we studied some classes of star covering spaces. Also, we studied some relations between star covering spaces and other well known spaces.

Moreover, we studied some topological properties on these spaces as: subspace, product, image and pre-image.

الملخص

بعض صفوف الفضاءات الغطائية النجمية

إعداد

فاطمة هشام ولدعلي

المشرف

أ.د. عدنان البصول

في هذه الرسالة قمنا بدراسة بعض صفوف الفضاءات الغطائية النجمية. ثم قمنا بدراسة العلاقة بين هذه الفضاءات وفضاءات معروفة سابقاً.

كذلك قمنا بدراسة أهم وأبرز الخصائص التوبولوجية لهذا النوع من الفضاءات، مثل: الفضاءات الجزئية، الجداء،... الخ.

GLOSSARY OF NOTATIONS

There are certain standard notation and terminologies used by mathematicians. In addition, we use a few special notations with less currency.

The following notations will be used.

$\mathbb{R}$	The real numbers
$\mathbb{Z}$	The integer numbers
$\mathbb{N}$	The natural numbers
$\mathbb{Q}$	The rational numbers
$\mathbb{I}$	The irrational numbers
$\lvert A \rvert$	The cardinality of a set A
$\bar{A}$	The closure of a set A
$\emptyset$	The empty set
$x \in A$	x is a member of a set A
$St(A, \mathcal{U})$	Star of the $A \subseteq X$ with respect to the family $\mathcal{U}$ of subsets of X
$ext(X)$	The extent of a space X

<u>KEY WORDS AND PHRASES</u>

Star-Lindelöf, star-compact, star-countable, star-finite, star-P, absolutely star compact, absolutely star- Lindelöf, strongly star-compact, strongly star-Lindelöf, weakly star-compact, neighborhood star-Lindelöf.

Chapter 1

Preliminaries

This chapter contains three sections; the first will introduce some basic and essential concepts, definitions and theorems in topology. That will be used in the sequent chapters. The second section will give us a brief history about star covering spaces and star covering properties.

1. 1. Introduction

Throughout this work, if nothing is said about the axioms of separation of a space X, then X is assumed to be a topological space on which no separation axioms are considered.

Given a topological space X, the family $\tau(X)$ is a topology for X; if $x \in X$ then $\tau(x) = \{U \in \tau(X): x \in U\}$. Our set-theoretic notation is standard; let (X, τ) be a topological space, let $\mathcal{A} = \{A_\alpha: \alpha \in \Delta\}$ be a family of subsets of X and $Y \subseteq X$. The family $\mathcal{A}$ covers Y if and only if $Y \subseteq \cup A_\alpha$, if Δ finite (or countable respectively) and $\mathcal{A}$

covers Y, then $\mathcal{A}$ is called finite (or countable respectively) cover of Y. If for each $\alpha \in \Delta, A_\alpha$ is open (or closed respectively) in X and $\mathcal{A}$ covers Y, then $\mathcal{A}$ is called an open (or closed respectively) cover of Y.

Throughout this thesis, for a cardinal $\mathcal{K}, \mathcal{K}^+$ denotes the smallest cardinal greater than $\mathcal{K}$. Let ω be the first infinite ordinal, ω_1 the first uncountable ordinal and $\mathfrak{c}$ the cardinality of continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals.

We write,

$\cup \mathcal{U}$ for $\bigcup_{U \in \mathcal{U}} U$, where $\mathcal{U}$ is some collection of sets;

$\mathcal{P}(X)$ for the power set of X;

We always consider $\mathbb{R}$ with the usual topology, unless otherwise stated.

The following are the well-known topologies on X:

1. The discrete topology is $\tau_{dis.} = \mathcal{P}(X)$.
2. The co-finite topology is $\tau_{cof.} = \{U \subseteq X : X \setminus U \ finite\} \cup \{\emptyset\}$.
3. The co-countable topology is $\tau_{coc.} = \{U \subseteq X : X \setminus U \ countable\} \cup \{\emptyset\}$.

We say that a set is clopen if it is both open and closed. Any open set $U \subseteq X$ with $x \in U$ is called a neighborhood of x. For any $x \in X$ the family of open sets $\mathcal{V}_x = \{V_\alpha \subseteq X : \forall\, \alpha \in \Delta,\ x \in V_\alpha\}$ is called a local (neighborhood) basis at the point x if for any open $U \subseteq X$ such that $x \in U$ there is an element $V \in \mathcal{V}_x$ such that $x \in V \subseteq U$. The product topology on $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$.

Recall some basic definitions that related to any topological space X. The following definitions will be useful in the next chapters we can find them in [10] and [25]. Any additional definitions that will be needed will present in its suitable place in this thesis.

Definition 1.1.1. Let (X, τ) be a topological space, a space X is called

- *First countable* if for any $x \in X$ there is a countable local base at x.
- *Second countable* if its topology has a countable base.
- Satisfying *the countable chain condition* if every family of non-empty disjoint open sets of X is countable. We say that X is CCC.
- *Separable* if it has a countable dense subset D of X.
- *Connected* iff we can't write the space X as union of two disjoint non-empty open sets.

Definition 1.1.2. (Separation axioms on X) Let (X, τ) be a topological space, a space X is said to be a:

- T_0-space iff for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subseteq X$ such that U contains one of x and y but not the other.
- T_1-space or Frechet iff for each $x, y \in X$ such that $x \neq y$ there are two sets $U, V \subseteq X$ such that $x \in U, y \in V$ and $x \notin V, y \notin U$ (which is, there is a neighborhood of each not containing the other).
- T_2-space or Hausdorff space iff for each $x, y \in X$ such that $x \neq y$ there exist two disjoint open sets $U, V \subseteq X$ such that $x \in U, y \in V$.
- *Urysohn* if whenever $x \neq y \in X$ there are two open sets U, V with $x \in U, y \in V$, and $\overline{U} \cap \overline{V} = \emptyset$

- *Regular* space iff for each $x \in X$ and each closed $C \subseteq X$ such that $x \notin C$, there exist two disjoint open sets $U, V \subseteq X$ such that $x \in U, C \subseteq V$. A regular T_1-space is called a T_3-space.
- *Normal* iff whenever A and B are disjoint non empty closed subsets of X, there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$. A normal T_1 space is called a T_4-space.

Note about the above definition:

- Obviously the properties T_0, T_1, T_2, T_3 are productive, hereditary, and topological properties. Discrete spaces are T_0 but indiscrete spaces with more than one point are not T_0.
- It is obvious that every T_1 space is T_0, every T_2 space is T_1 and every T_3 space is T_2.
- The properties T_4 and *normality* are both topological properties but in general are not product preserving. Every T_4 space is clearly a T_3 space. But the *normal* spaces need not be *regular.*

Definition 1.1.3. Let (X, τ_X) and (Y, τ_Y) be topological spaces, a map $f: X \to Y$ is called

- *Continuous* if the inverse-image of any open set in Y is open in X (or equivalently, the inverse-image of every closed set in Y is closed in X).
- *Open* (*closed)* if the image of every open set in X is open (closed) in Y.

Definition 1.1.4. [25]. Let (X, τ) be a topological space, a *cover (or covering)* of a space X is a collection $\mathcal{A}$ of subsets of X whose union is all of X. A *subcover* of a cover

$\mathcal{A}$ is a subcollection $\acute{\mathcal{A}}$ of $\mathcal{A}$ which is a cover. An *open cover* of X is a cover consisting of open sets, and other adjectives applying to subsets of X apply similarly to covers.

Definition 1.1.5. Let (X, τ) be a topological space, a space X is called

- *Compact* iff every open cover of X has a finite subcover.
- *Countably compact* iff every countable open cover of X has a finite subcover.
- σ*-compact* iff X can be written as a countable union of compact sets.
- *Pseudocompact* if each continuous real-valued function on X is bounded.
- *Lindelöf* iff every open cover of X has a countable subcover.

Definition 1.1.6. Let (X, τ) be a topological space,$x \in X$, and $A \subseteq X$. Then x is a cluster point (or accumulation point, or limit-point) of Aif and only if every open set contains at least one point of A different from x.

Definition 1.1.7. Let (X, τ) be a topological space, the space X has the Bolzano-Weirestrass property if and only if every infinite subset of X has a cluster point belonging to X.

Theorem 1.1.8. *The space X is countably compact if and only if each sequence in X accumulates to a point in X.*

Theorem 1.1.9. *Every countably compact space X has the Bolzano-Weirestrass property.*

Theorem 1.1.10. *Let X be a T_1-space which has the Bolzano-Weirestrass property. Then X is countably compact.*

Definition 1.1.11 [25]. A subset of a topological space X *is a* G_δ if and only if it is a countable intersection of open sets and an F_σ if and only if it is a countable union of closed sets.

Definition 1.1.12 [25] Let (X, τ_X) and (Y, τ_Y) be topological spaces, a map $f: X \to Y$ is called *Perfect* if f is continuous, closed, and $f^{-1}(y)$ is compact in Y for each $y \in Y$.

One of the basic theorems in Real Analysis, the Heine-Borel Theorem, states that every closed bounded interval of the real line is compact.

Theorem 1.1.13 [10]. *(The Heine-Borel Theorem) A subset C of $\mathbb{R}$ is compact iff it is closed and bounded.*

Definition 1.1.14 [25]. (The one-point-compactification or Alexandroff compactification). Let X be a locally compact, noncompact Hausdorff space, p a point not in X. Let $X^* = X \cup \{p\}$, and let the basic neighborhoods of p be the sets of the form $\{p\} \cup (X \backslash L)$, where L is a compact set in X. Neighborhoods of points in X are unchanged in X^*. Clearly X^* is compact and X is open and dense in X^*. Moreover, X^* is Hausdorff. We will call X^* *the one-point-compactification (Alexandroff compactification)* of X.

Definition 1.1.15 [25]. A collection $\mathcal{U}$ of subsets of X is *locally-finite (locally-countable)* if and only if each $x \in X$ has a neighborhood meeting only finitely (countably) many $U \in \mathcal{U}$. We call $\mathcal{U}$ *point-finite (respectively, point-countable)* if and only if each $x \in X$ belongs to only finitely (countably) many $U \in \mathcal{U}$.

Definition 1.1.16 [25]. A collection $\mathcal{U}$ of subsets of X is *discrete* if and only if each $x \in X$ has a neighborhood meeting at most one element of $\mathcal{U}$. Clearly every discrete collection of sets is locally-finite.

Definition 1.1.17 [25]. Let (X, τ) be a topological space, a space X is called *completely regular space* if given any closed subset $F \subseteq X$ and any point $x \notin F$, then there is a continuous real-valued function such that $f(x) = 0$ and $f(y) = 1, \forall\, y \in Y$. (i.e. x and F can be separated by a continuous function). X is *Tychonoff or completely - T_3* if it is both completely regular and Hausdorff.

We need the following theorem and definition in chapter 3 and in chapter 4, any undefined terms can be founded in [25].

Theorem 1.1.18 [25]. *The following statements are equivalent:*

a) Axiom of choice: if $\{A_\alpha : \alpha \in \Delta\}$ is an indexed family of nonempty pairwise disjoint sets, there is a set $B \subseteq \bigcup A_\alpha$ such that $B \cap A_\alpha$ is exactly one element for each $\alpha \in \Delta$.

b) Zorn's lemma: If each chain (linearly ordered set) in a nonempty partially ordered set A has an upper bound, then A has a maximal element.

Definition 1.1.19 [25]. *The ordinal space* $\omega_1 + 1 = [0, \omega_1]$ where ω_1 is the first uncountable ordinal number. The element ω_1 is the limit-point of the subset $[0, \omega_1)$. Where the subspace $\omega_1 = [0, \omega_1)$.

1. 2. A Brief History about Star Covering

If X is a topological space and $\mathcal{A}$ is a family of subsets of X; then the star of Y with respect to $\mathcal{A}$ is the set

$$St(Y,\mathcal{A}) = \cup \{A \in \mathcal{A} : Y \cap A \neq \emptyset\}, \text{ for any } Y \subseteq X.$$

Inductively, one can define $St^{n+1}(Y,\mathcal{A}) = St(St^{n}(Y,\mathcal{A}),\mathcal{A})$ for $n \in \mathbb{N}$. For instance, $St^{2}(Y,\mathcal{A}) = St(St^{1}(Y,\mathcal{A}),\mathcal{A})$

The concepts in the following definitions are covered in details in [8] and [9] using the following terminology:

Definition 1.2.1 [1]. (Iterated stars). Suppose $\mathcal{A}$ is a family of subsets of X and $Y \subseteq X$. For $n \in \mathbb{N}$, we define $St^{n+1}(Y,\mathcal{A}) = St(St^{n}(Y,\mathcal{A}),\mathcal{A})$ where $St^{0}(Y,\mathcal{A}) = Y$.

Also, for a space X, a subset $Y \subseteq X$ and a family of subsets $\mathcal{A}$,

$$St^{\infty}(Y,\mathcal{A}) = \bigcup_{n=1}^{\infty} St^{n}(Y,\mathcal{A})$$

Let $\mathcal{U} = \{U_{\alpha} : \alpha \in \Delta\}$ be a cover of a space X; the star of the singleton $\{x\}$ with respect to $\mathcal{U}$ is called *the star of the point x with respect to* $\mathcal{U}$ and is denoted by $St(x,\mathcal{U})$.

Definition 1.2.2 [4]. *The extent of a space X;* denoted by $ext(X)$; is the smallest infinite cardinal number k such that every closed and discrete subspace of X has cardinality at most k.(i.e. if A is a closed and discrete subset of X, then $|A| \leq k$. For instance; a space has countable extent provided if every closed discrete subspace is at most countable. (i.e. $ext(X) = \aleph_0$, there are no uncountable closed and discrete subset).

It is well known that Lindelöf spaces are countably compact spaces and having countable extent.

All topics and concepts related to star covering spaces was covered in Matveev's survey [13].

1. 3. Historical Background

Most of the researchers of star covering worked in the next concept, they gave the subset $A \subseteq X$ several properties and all of their attention was around it!

"For every open cover $\mathcal{U}$ of X there exists a subset $A \subseteq X$ such that $St(A, \mathcal{U}) = X$"

The study of covering properties could be dated to 1970s, a systematic investigation on them was done be van Douwen in 1990s [9]. One of the most effective studies in our subject; is Song in his paper [23], and Alas in her papers [2] and [3].

The study of covering properties is one of the major topics in general topology. It seems that the study of star covering properties of topological spaces does not have a very long history; in the last forty years, a new type of covering properties, namely star covering properties, has attracted many general topologists. A relatively comprehensive survey is Matveev [13]. We shall give some basic and important historical stations on this topic.

Fleischman [11] who first used the term "star-compact" in 1970 and assumed the subset A to be finite, also he proved that every countably compact space is star-compact. After that star-compactness and its generalizations were further considered by Ikenga, who assumed A to be countable in 1983 and called this property "star-*Lindelöf*", after that and in 1987 Ikenga studied n-star- *Lindelöf* spaces under the name $\omega - n - star - Lindel\ddot{o}f$ spaces [7].

During 1980s, Matveev and Sarkhel published many papers around the subject. But the first systematic investigation on star covering properties was done by van Douwen in 1991 [9].

In 2006, Song [21] put A to be Lindelöf and this property was called $\mathcal{L} - star - Lindel\ddot{o}f$.

In 2007, Song [20] gave A compact and this property was called $\mathcal{K} - star - compact$.

In 2008, Song [19] considered A countably compact and this property was called $\mathcal{C} - star - compact$.

On the other hand, the researchers change previous concepts so that for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$, there exists a subset $A \subseteq D$ such that $St(A, \mathcal{U}) = X$. And so they focused on the subset $A \subseteq X$ and change its properties, for example:

In 1994, Matveev [13] suppose the subset A under the assumption of the previous definition to be finite and this property was called absolutely countably compact (acc), then and in 1997, he grant the set A discrete and called this property $(\omega_a) - space$, in the same year he donate the set A closed and discrete and called this property (a)-space. But the relatively comprehensive survey [13] contained the known results tell that time.

In 1998, Bonanzinga [4] considered a set A to be countable and this property was called a-star- *Lindelöf.*

In 2011; a simple logical and general terminology was appeared instead of previous concept of the star covering [3]. The new concept due to van Mill who defined the term and then Ofelia Alas deals with this term in an extensively way, with the subset A as a subspace and the whole space takes his name from the property that carried by the subspace A, the new approach made a difference from other definitions. It is simple, direct and logical concept.

Generally and considering Alas approach, "whenever P is a topological property, we say that a topological space is star-P if whenever $\mathcal{U}$ is an open cover of X there is a subspace $A \subseteq X$ with property P such that $X = St(A, \mathcal{U})$". In [3] Alas studied the relationships of star-P properties for $P \epsilon \{Lindel\ddot{o}f, \sigma - compact, countable\}$ with other Lindelöf type properties.

In the next chapters, we will try to understand the relation between Alas definitions and the previous concepts, and to study her approach in some details.

Chapter 2

Classes Defined by Stars and Related Spaces

(Absolutely, Strongly) concepts

To avoid any ambiguity, in this chapter we shall study some topological properties in details. First of all, there are some basic definitions that studied by many topologist. Also, and according to the reference [16], Rawashdeh [16] in her thesis focused on *absolutely countably compact space (acc)*, *countably (acc) space*, *absolutely star- Lindelöf spaces (a-star-Lindelöf)* and *(a)-space*. While we will focus on *strongly star-compact space*, *strongly star-Lindelöf* space in this chapter.

2. 1. Basic Definitions and Implications

In 1967, Aquaro studied the problem when every point-countable open cover of a space X has a countable subcover [7]. He considered the following property:

"Any discrete family of nonempty closed sets in X is countable" … (*)

Aquaro observed that all countably compact spaces and all Lindelöf spaces have the property (*), and if a space X has such property then every point countable open cover of X has a countable subcover.

Definition 2.1.1 [11]. (Star-compact). A space X is said to have ***star-compact*** property if for every open cover $\mathcal{U}$ of X, their exists a finite subset $F \subseteq X$ such that $St(F, \mathcal{U}) = X$.

The concept of star-compact, in the class of Hausdorff space coincide with countably compact (henceforward denoted cc; the proof of this fact is given in Theorem 2.1.2 and Theorem 2.1.3), was firstly introduced in 1970 by Fleischman [11]. The following two theorems show that in the class of Hausdorff spaces, countably compactness and star-compactness are equivalent.

Theorem 2.1.2 [9] *Every countably compact space is star-compact space.*

Proof. Suppose on the contrary, that X is not star-compact. So, there exists an open cover $\mathcal{U}$ of X and for every finite subset F of X, then $St(F, \mathcal{U}) \neq X$. Fix any $x_0 \in X$ and, inductively, fix $x_n \in (St(\{x_0, x_1, \dots, x_{n-1}\}, \mathcal{U}))^c$ for $n > 0$. Let $M = \{x_n : n \in \mathbb{N}\}$ and $\mathcal{V} = \{St(x_n, \mathcal{U}) : n \in \mathbb{N}\}$. Note that by the choice of x_n, every member of $\mathcal{V}$ contains exactly one element of M, no finite subfamily of $\mathcal{V}$ will cover M. Let $r \in \overline{M}$, then there exist $U \in \mathcal{U}$ such that $r \in U$. As $U \cap \overline{M} \neq \emptyset$, $U \cap M \neq \emptyset$ and hence $r \in St(x_n, \mathcal{U})$ for some n. Therefore, $\mathcal{V}$ is a countable covering of $\overline{M}$ by sets open in X. $\overline{M}$ is countably compact. Therefore, there exists a finite subfamily of $\mathcal{V}$ which covers $\overline{M}$ and hence M. This contradicts our last observation about $\mathcal{V}$.■

Theorem 2.1.3 [9]. *Star-compact Hausdorff spaces are countably compact.*

Proof. Let X be a star-compact Hausdorff space which is not countably compact. Then X would contain an infinite subset $D = \{x_n : n \in \mathbb{N}\}$ with no cluster points, D is closed and discrete in the subspace topology, hence for each $n \in \mathbb{N}$ there exists an open set U_n such that $U_n \cap D = \{x_n\}$. For every $m \in \mathbb{N}$, define $B_m = \{x_n \in D: 2^m \leq n < 2^{m+1}\}$, so $|B_m| = 2^m$. Since B_m is finite and X is Hausdorff, there exist disjoint open sets V_n such that $x_n \in V_n$ for $2^m \leq n < 2^{m+1}$. Next, setting $\mathcal{V}_m = \{U_n \cap V_n: 2^m \leq n < 2^{m+1}\}$ gives a collection of pairwise disjoint open subsets of X such that $(U_n \cap V_n) \cap D = \{x_n\}$. Define $\mathcal{V} = \{D^c\} \cup \bigcup_{m \in \mathbb{N}} \mathcal{V}_m$. Evidently, $\mathcal{V}$ is an open cover of X. Let F be any finite subset of X, with $|F| = M$, say. Then $|F| < 2^m = |\mathcal{V}_M|$. So, for some $2^M \leq m < 2^{M+1} - 1$, $(U_m \cap V_m) \cap F = \emptyset$. But $U_m \cap V_m$ is the only member of $\mathcal{V}$ that contains x_m. Thus, $x_m \notin St(F, \mathcal{V})$. But F was an arbitrary finite subset of X, so X is not star-compact. This is a contradiction.■

Similarly to the definition of the term of star-compact, the term of star- Lindelöf was introduced by Ikenga in 1983 (see [12]).

Definition 2.1.4. [12]. (Star-Lindelöf). A space X is said to have ***star- Lindelöf*** property if for every open cover $\mathcal{U}$ of X, their exists a countable subset $C \subseteq X$ such that $St(C, \mathcal{U}) = X$.

Clearly, all Lindelöf spaces are star-Lindelöf, also all separable spaces are star-Lindelöf spaces, all compact spaces are star-compact spaces, and all star-compact spaces are star- Lindelöf. Summing up the above theorems with some arrangement of the definitions, we can express this as the following diagram, where the implications (1)-(6) hold for arbitrary spaces.

$$\begin{array}{ccc} & \text{Countably Compact} & \\ & \downarrow 1 & \\ \text{Compact} & \underset{2}{\rightarrow} & \text{Star compact} \\ \downarrow 3 & & \downarrow 4 \\ \text{Lindelof} & \underset{5}{\rightarrow} & \text{Star}-\text{lindelof} \\ & & \uparrow 6 \\ & & \text{Separable} \end{array}$$

Diagram 2.1

We will give examples that show that the implications (1)-(6) in the above diagram cannot be reversed.

Moore plane Γ (also known as the Niemytzki's tangent disk space, see [24, Example 82]) shows that the converse of implications (4) and (5) do not hold. Γ is a star-Lindelöf space (since separable) which is neither Lindelöf nor star-compact.

The next example shows that the converse of implications (1), (2), (3) and (6) do not hold.

Example 2.1.5 [9] *Consider the topological space* $X = (\mathbb{R}, \tau_{coc.})$. *Clearly the space is not separable and not compact.*

X is not countably compact space, indeed it is not compact, the compact subsets of $\mathbb{R}$ *are only the finite one. Let* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ *be a countable open cover of* X *such that* $V_n = (\mathbb{R} \setminus \mathbb{N}) \cup \{n\} \in \tau_{coc.}, \forall\, n \in \mathbb{N}$. *This open cover* $\mathcal{V}$ *does not contain a finite subcover since* $V_n \cap \mathbb{N} = \{n\}, \forall n \in \mathbb{N}$. *Which means that X is not countably compact space.*

Clearly the real line with the cocountable topology is not Hausdorff space.

To prove that X is star-compact space, let $\mathcal{U}$ be an open cover of $\mathbb{R}$ and fix any nonempty $U_0 \in \mathcal{U}$.

Case 1. If $U_0 = \mathbb{R}$, then let $F = \{x_0\}$; clearly, $St(F, \mathcal{U}) = \mathbb{R}$.

Case 2. If $U_0 \neq \mathbb{R}$, then $\mathbb{R} \backslash U_0 = \{x_n : n \in \mathbb{N}\}$. For each x_n fix some $U_n \in \mathcal{U}$ with $x_n \in U_n$. Then $\mathcal{V} = \{U_n : n \in \mathbb{N}\} \cup \{U_0\} \subseteq \mathcal{U}$ is an open cover of $\mathbb{R}$.

$$X \backslash \bigcap_{n=0}^{\infty} U_n = \bigcup_{n=0}^{\infty} (X \backslash U_n);$$

This is countable. Therefore, $\bigcap_{n=0}^{\infty} U_n$ is nonempty, because X is uncountable, select any $x \in \bigcap_{n=0}^{\infty} U_n$, let $F = \{x\}$ then

$$X = \bigcup_{n=0}^{\infty} U_n = St(x, \mathcal{V}) \subseteq St(F, \mathcal{U}) \subseteq X.$$

Hence X is star-compact.

Theorem 2.1.6. [11]. *Star-compactness implies pseudocompactness.*

Proof. Let X be a space which is star-compact and let f be a continuous real-valued function on X. For each integer n, let $\mathcal{U}_n = \{x \in X : n - 1 < f(x) < n + 1\} = \{f^{-1}\big((n - 1, n + 1)\big) : n \in \mathbb{N}$. Then $\mathcal{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ is a point-finite open covering of X, so that $\mathcal{U}$ has a finite subcover $\{\mathcal{U}_{n_1}, \mathcal{U}_{n_2}, \dots, \mathcal{U}_{n_k}\}$. Let $M = \max_{1 \le j \le k} |n_j| + 1$. For each $x \in X$, there is some j, $1 \le j \le k$, with $x \in \mathcal{U}_{n_j}$. Thus $|f(x)| < |n_j| + 1 < M$.■

The topological space **Irrational Slope Topology,** see [24, Example 75] is an example that the converse of the above theorem is not true in general. Indeed Irrational Slope Topology is $T_2 - space$ which is not countably compact. To prove that consider $X = \{(x,y) : y \geq 0, x, y \in \mathbb{Q}\}$ and fix some irrational number $\theta \in \mathbb{I}$. The irrational slope topology τ on X is generated by $\epsilon -$nneighborhoods of the form $N_\epsilon\big((x,y)\big) = \{(x,y)\} \cup B_\epsilon(x + \frac{y}{\theta}) \cup B_\epsilon(x - \frac{y}{\theta})$ where $B_\epsilon(p) = \{r \in \mathbb{Q}: |r - p| < \epsilon\}$, each $N_\epsilon((x,y))$ consists of $\{(x,y)\}$ plus two intervals on the rational $x - axis$ centered at two irrational points $x \mp \frac{y}{\theta}$; the lines joining these points to (x,y) have slope $\mp\theta$. The topological space (X,τ) is Hausdorff since θ is irrational, for no two points in X can lie on a line with slope θ, and if one point of X lies on a line with slope θ, no other points of X can lie on the line of slope-θ which intersects the original line at its intersection with the $x - axis$. Thus any two distinct points in X must project (along lines with slope $\mp\theta$) onto distinct pairs of irrational points on the $x - axis$, which have distinct neighborhoods. The irrational slope space is pseudocompact, since every real-valued continuous function f on (X,τ) is constant, for if f were not constant, $f(X)$ would contain two disjoint open sets with disjoint closures. The inverse images would then be disjoint sets with disjoint closures, which is impossible. Thus X is pseudocompact. Finally X does not satisfy Bolzano-Weierstrass property, since the sequence of integers on the $x - axis$ has no limit point, which implies that X is not countably compact space.

Definition 2.1.7. [18]. A space X is said to have ***countably star-compact*** property if for every countable open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F, \mathcal{U}) = X$.

It is easy to show that every star-compact is countably star-compact and all countably compact spaces are countably star-compact.

Theorem 2.1.8. [18] *Every countably star-compact space X is pseudocompact.*

Proof. Let X be a space which is countably star-compact and let f be a continuous real-valued function on X. For each $n \in \mathbb{Z}$, let $\mathcal{U}_n = \{x \in X : n-1 < f(x) < n+1\}$. Then $\mathcal{U} = \{\mathcal{U}_n : n \in \mathbb{Z}\}$ is a countable open covering of X. Since X is countably star-compact, there exist a finite subset B of X such that $St(B, \mathcal{U}) = X$. Since $\mathcal{U}$ is point-finite, the set $\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$ is finite, say $\{\mathcal{U}_{n_1}, \mathcal{U}_{n_2}, \ldots, \mathcal{U}_{n_k}\}$. If we put $M = max\{|n_j| + 1 : i = 1,2, \ldots, k\}, then$. For each $x \in X$, $|f(x)| \leq M$. Hence, every continuous real-valued function on X is bounded, which means that X is pseudocompact.■

Proposition 2.1.9. [18]. *Every Hausdorff space is countably compact iff it is countably star-compact*

Proof. The proof of this proposition is similar to the proofs of Theorem 2.1.2 and Theorem 2.1.3.

Y.- K. Song in his papers [19], [20] and [21] gives some specific definitions that have the way for O. Alas papers [2] and [3] have in her approach about star covering spaces.

Definition 2.1.10 [20] A space *X* is said to have $\mathcal{K} - star - compact$ property if for every open cover $\mathcal{U}$ of *X,* there exists a compact subset *K* of *X* such that $St(K, \mathcal{U}) = X$.

Definition 2.1.11. [21]. A space *X* is said to have $\mathcal{L} - star - lindelof$ property if for every open cover $\mathcal{U}$ of *X,* there exists a Lindelöf subset *L* of *X* such that $St(L, \mathcal{U}) = X$.

Rawashdeh [16] suggested the name of " $\mathcal{L}-star-lindelof$ " instead of $\mathcal{L}-star-compact$ which was mentioned in [21], in fact, her suggestion was based on a Lindelöf set not on a compact set.

Another definition related to the above two definitions was stated in [19]

Definition 2.1.12. [19]. A space X is said to have $\mathcal{C}-star-compact$ property if for every open cover $\mathcal{U}$ of X, there exists a countably compact subset C of X such that $St(C,\mathcal{U})=X$.

From above definitions it is not difficult to see that every

Every star-compact is $\mathcal{K}-star-compact$.

Every star-Lindelöf is $\mathcal{L}-star-lindelof$.

Every $\mathcal{K}-star-compact$ is $\mathcal{L}-star-lindelof$.

Every $\mathcal{K}-star-compact$ is $\mathcal{C}-star-lindelof$.

(Attention, definitions for star- compact and star-Lindelöf will be changed a little bit in section 2. 2!)

2. 2. On Strongly Star Spaces

Some authors (see [6], [9], [13],[23]) used the term "Strongly star-compact" instead of the previous definition of star-compact, and the term "Strongly star-Lindelöf" instead of the previous definition of star-Lindelöf. Rewrite the definitions for both star-compact and star-Lindelöf, like the following:

Definition 2.2.1. [23]. A space X is said to be *star-compact (resp., star-Lindelöf)* if for every open cover $\mathcal{U}$ *of* X*, there exists a finite(resp., countable) subset* $\mathcal{V}$ *of* $\mathcal{U}$ *such that* $St(\bigcup\mathcal{V}, \mathcal{U}) = X$.

Clearly, in the case such that the above definition holds,

$$\text{Strongly star} - \text{compact space} \rightarrow \text{Strongly star} - \text{Lindelöf space}$$

$$\text{Strongly star} - \text{compact space} \rightarrow \text{Star} - \text{compact space}$$

$$\text{Strongly star} - \text{Lindelöf space} \rightarrow \text{Star} - \text{Lindelöf space}$$

Based on above definition every countably compact space is strongly star-compact, and every Hausdorff strongly star-compact space is countably compact (see Theorem 2.1.2 and Theorem 2.1.3).

Definition 2.2.2. [9]. A space X is said to be *n-star-compact (resp., n-star-Lindelöf)* if for every open cover $\mathcal{U}$ of X, there is some finite (resp., countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $St^n(\bigcup\mathcal{V}, \mathcal{U}) = X$.

Definition 2.2.3. [9]. A space X is said to be *strongly n-star-compact (resp., strongly n-star-Lindelöf)* if for every open cover $\mathcal{U}$ of X, there is some finite (resp., countable) subset A of X such that $St^n(A, \mathcal{U}) = X$.

It is easy to see that if X is strongly n-star-compact, then X is n-star-compact, and if X is n-star-compact, then X is strongly $n+1$-star-compact. A similar hierarchy holds for the star- Lindelöf properties. For T_2 –spaces, strongly 1-star-compact are identical with countably compact.

Theorem 2.2.4. [9]. *If X is strongly n-star-compact, then X is n-star-compact.*

Proof. Let $\mathcal{U}$ be an open cover of X, by hypothesis there is a finite subset $A \subseteq X$ such that $St^n(A, \mathcal{U}) = X$. For each $a \in A$ select some open $U_a \in \mathcal{U}$ such that $a \in U_a$. Let $\acute{\mathcal{U}} = \{U_a : a \in A\}$. So $\acute{\mathcal{U}}$ is a finite subset of $\mathcal{U}$ and

$$X = St^n(A, \mathcal{U}) \subseteq St^n(\cup \acute{\mathcal{U}}, \mathcal{U}) \subseteq X. \blacksquare$$

Theorem 2.2.5. [9]. *If X is n-star-compact, then X is strongly n+1-star-compact.*

Proof. (Similar to proof of Theorem 2.2.4).■

In the following definition van Douwen et al. [9] considers the extension to *n-star-compact, ω-star-compact* and *ω-star- Lindelöf properties*.

Definition 2.2.6. [9]. A space X is said to be *ω-star-compact (resp., ω-star-Lindelöf)* if for every open cover $\mathcal{U}$ of X, there is some $n \in \mathbb{N}$ and some finite (resp., countable) subset A of X such that $St^n(A, \mathcal{U}) = X$.

The concept of *strong star-compactness* was introduced by Fleischman in [11]. Later Sarkhel in [17] extended his work, and defined the terms of *n-star-compactness* and *ω-star-compactness*. After that Matveev defined the strong *n-star-compactness.* While Ikenga studied the *strong star- Lindelöfness* (see [9])

It is clear that van Douwen takes the following as an alternative, though equivalent, definition of ω-star-compact (resp., ω-star-Lindelöf)

Definition 2.2.7. [9]. A space X is said to be *ω-star-compact (resp., ω-star-Lindelöf)* if for every open cover $\mathcal{U}$ of X, there is some $n \in \mathbb{N}$ and some finite (resp., countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $St^n(\cup\mathcal{V}, \mathcal{U}) = X$.

Indeed, these covering properties are all weaker than the compactness. In fact, they are between countable compactness and pseudocompactness, as diagram 2.2 below shows.

Van Douwen proved (see [9] and in a similar way to the proof of Theorem 2.1.2 and Theorem 2.1.3) that every countably compact space is strongly 1-star-compact, and every strongly 1-star-compact Hausdorff space is countably compact. Next theorem shows that for Tychonoff spaces, ω-star-compact spaces are pseudocompact.

Theorem 2.2.8. [9]. *Every ω-star-compact space X is pseudocompact space.*

Proof. Suppose that X is *ω-star-compact* and that $f: X \to \mathbb{R}$ is continuous. Define $\mathcal{U} = \{f^{-1}(k, k+2): k \in \mathbb{Z}\}$. Then $\mathcal{U}$ is an open cover of X and for some $n \in \mathbb{N}$ and for some finite $\mathcal{V} \subseteq \mathcal{U}$, $St^n(\cup\mathcal{V}, \mathcal{U}) = X$. Let $M = \max\{k+2: f^{-1}(k, k+2) \in \mathcal{V}\}$ and $m = \min\{k : f^{-1}(k, k+2) \in \mathcal{V}\}$. It is now clear that $f(X) \subseteq (m - 2n, M + 2n)$. If $x \in X$, then for $1 \le j \le n$ there is $f^{-1}(k_j, k_j + 2) \in \mathcal{U}$ such that $x \in f^{-1}(k_n, k_n + 2)$ with $f^{-1}(k_j, k_j + 2) \cap f^{-1}(k_{j+1}, k_{j+1} + 2) \neq \emptyset$ and $f^{-1}(k_1, k_1 + 2) \cap \cup\mathcal{V} \neq \emptyset$. By contradiction, $f(\cup\mathcal{V}) \subseteq (m, M)$. By induction, we can shows that $f(x) \in (m - 2n, M + 2n)$, which is required.■

The above results complete diagram 2.2 and give us simple connections between the previous definitions.

$countably\ compact$

$\downarrow \quad \uparrow \ T_2$

$strongly\ 1-star-compact$

$\downarrow$

$1-star-compact$

$\downarrow$

$strongly\ 2-star-compact$

$\downarrow$

$2-star-compact$

$\downarrow$

.
.
.

$\downarrow$

$strongly\ n-star-compact$

$\downarrow$

$n-star-compact$

$\downarrow$

$strongly\ n+1-star-compact$

$\downarrow$

.
.
.

$\downarrow$

$\omega-star-compact$

$\downarrow\uparrow$

$Psudocompact$

Diagram 2.2

2. 3. On Absolutely Countably Compact Spaces (acc) and Absolutely Star-Lindelöf spaces (a-star-Lindelöf)

Rawashdeh in her thesis [16] discussed this topological property extensively in Chapter 3 of her thesis, M. Matveev [13] defined a topological property for the first time in his paper that called "absolutely countably compact (acc)" which is stronger than star-compact and with Hausdorff it is stronger than countably compact.

Definition 2.3.1. [13]. A topological space X is *absolutely countably compact (acc)* provided that for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$, there exists a finite subset $F \subseteq D$ such that $St(F, \mathcal{U}) = X$.

Theorem 2.3.2 [13]. *Every compact space is (acc), and every (acc) space is star-compact space.*

Proof. Direct from the definitions of each term. ■

Y.-K. Song [18] gave a counter example that shows that the converse of the first implication (every compact space is (*acc*)) does not hold in [18, Example 2.4], which ensures that there exist an acc T_1-space which is not countably compact. M. Matveev, gives us an example of a star-compact space which is not acc space, that is the converse of the second implication does not hold.

Next, we give the definition of a strongly subspace of a space X, which will be needed in order to define Strongly (acc)

Definition 2.3.3. [22]. A subspace Y of a space X is *strongly star-compact* in X if for every open cover $\mathcal{U}$ of X, there exists a finite subset $F \subseteq Y$ such that $Y \subseteq St(F, \mathcal{U})$.

Let X be a space and Y be a subspace of X. It is clear that if Y is strongly star-compact in X, then Y is star-compact in X, and if Y is (acc) in X, then Y is star-compact in X.

Definition 2.3.4. [22]. A subspace Y of a space X is *strongly absolutely countably compact* in X if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq Y$ there exists a finite subset $F \subseteq D$ such that $Y \subseteq St(F, \mathcal{U})$.

In 2007, Song [18] replaced the open cover $\mathcal{U}$ from the definition of acc by countable open cover and he got the following definition.

Definition 2.3.5. [18]. A topological space X is *countably absolutely countably compact (= countably acc)* provided that for every countable open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$, there exists a finite subset $F \subseteq D$ such that $St(F, \mathcal{U}) = X$.

Obviously, every acc is countably acc and every countably acc is countably star-compact.

$$acc \underset{1}{\rightarrow} countably\ acc \underset{2}{\rightarrow} countably\ star - compact$$

In [18, Example 2.6 and Example 2.5], show that the converse of implications (1) and (2) , respectively, do not hold in general.

Theorem 2.3.6. [18]. *Every countably compact space is countably acc.*

Proof. Let X be a countably compact space, $\mathcal{U}$ be a countable open cover of X and D be a dense subspace of X. Then, there exists a finite subcover, $\{U_1, U_2, \dots, U_n\}$

of $\mathcal{U}$. Pick a point $x_i \in U_i \cap D$ for $i = 1,2,\dots,n$. Then, $St(\{x_1, x_2, \dots, x_n\}, \mathcal{U}) = X$, which shows that X is countably acc.■

Now, if we replace the finite set F in the definition of acc by a countable set, we get Bonanzinga new topological property "Absolutely Star-Lindelöf (a-star-Lindelöf)" which is weaker than acc and stronger than star-Lindelöf.

Definition 2.3.7. [4]. A topological space X is *absolutely star-Lindelöf (a-star-Lindelöf)* provided that for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$, there exists a countable subset $C \subseteq D$ such that $St(C, \mathcal{U}) = X$.

Theorem 2.3.8 [4]. *Every Lindelöf space is absolutely star- Lindelöf (a-star-Lindelöf), and every (a-star- Lindelöf) space is star- Lindelöf space.*

Proof. Direct from the definitions of each term.■

Theorem 2.3.9 [4]. Every (acc) space is an absolutely star-Lindelöf.

Proof. Direct from the definitions of each term.■

We Know that being (acc) gives the star- Lindelöf, in the next proposition we show that in countably compact spaces they are equivalent.

Proposition 2.3.10. [4]. *If a countably compact space X is a star-Lindelöf, then X is acc.*

Proof. Let X be a countably compact, a-star-Lindelöf space. Let $\mathcal{U}$ be an open cover of X and let D be any dense subspace in X, then there exists a countable subset $C \subseteq D$ such that $St(C, \mathcal{U}) = X$, then the family $\{St(x, \mathcal{U}) : x \in C\}$ is a countable open cover of X. Since X is countably compact, this cover has a finite subcover and then X is acc. ■

Theorem 2.3.2, Theorem 2.3.8, and Theorem 2.3.9 gives us the following diagram

$$\begin{array}{ccccc} compact & \rightarrow & acc & \rightarrow & star-compact \\ \downarrow & & \downarrow & & \downarrow \\ lindelof & \rightarrow & a-star-lindelof & \rightarrow & star-lindelof \end{array}$$

$\boldsymbol{Diagram\ 2.3}$

Chapter 3

On Weakly Star-Compact spaces and Neighborhood star-Lindelöf Spaces

In this chapter we consider previous definitions mentioned in section 2.3 of chapter 2.

3. 1. Weakly Star-Compact Spaces and Neighborhood Star-Lindelöf Spaces and their relations

A star covering property which is equivalent to countable compactness for regular spaces and weaker than countable compactness for Hausdorff spaces is introduced firstly by Bonanzinga et al. [5] and called "Weak Star-Compact" which is equivalent to countable compactness for regular spaces (in fact, for Urysohn spaces).

Theorem 3.1.1. [25]. *Every Regular T_1-space is Urysohn.*

Proof. Suppose X a regular T_1-space. Take $x, y \in X$ two different points, consider $(X\backslash\{y\})$ as a neighborhood of the point x , there exists U open subset of X such that $x \in U \subseteq \overline{U} \subseteq (X\backslash\{y\})$. Now consider $X\backslash\overline{U}$ as a neighborhood of the point y and from regularity, there exists V open subset of X such that $y \in V \subseteq \bar{V} \subseteq (X \setminus \bar{U})$. We get that $\bar{U} \cap \bar{V} = \emptyset$. Which means that X is Urysohn.■

Definition 3.1.2. [5]. A space X is *weakly star-compact* if for every open cover $\mathcal{U}$ of X, there is a finite subset $F \subseteq X$ such that for every open $O \supseteq F$, $St(O, \mathcal{U}) = X$.

In a similar way, Bonanzinga et al. defined "Neighborhood star-Lindelöf" property in [6], and in 2013, Y.- K. Song [23] investigate the relationship between neighborhood star-Lindelöf and related spaces.

Definition 3.1.3. [23]. A space X is said to be *weakly star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset $A \subseteq X$ such that for every open $O \supseteq A$, $St(O, \mathcal{U}) = X$.

From the definitions, it is clear that every weakly star-compact space is neighborhood star-Lindelöf, every strongly star-Lindelöf space is neighborhood star-Lindelöf, and, every neighborhood star-Lindelöf is star-Lindelöf. This gives us the following diagram.

$$\begin{array}{ccc} Weakly\ star-compact & & Strongly\ star-Lindelof \\ 1\downarrow & & 2\downarrow \\ & Neighborhood\ star-Lindelof & \\ & 3\downarrow & \\ & star-Lindelof & \end{array}$$

Diagram 3.1

Y.- K. Song [23] give an examples shows that the converse the implications (1) and (3) of the above diagram do not hold in general, Example 3.1.5 illustrate that the converse of (1) does not hold for the Tychonoff spaces and [23, Example 2.4] illustrates that the converse of the implication (3) does not hold. While Bonanzinga et al. mentioned in [6] that there exists a Urysohn neighborhood star-Lindelöf space that is not strongly star-Lindelöf.

Example 3.1.4. [23] *There exists a Tychonoff neighborhood star-Lindelöf space X that is not weakly star-compact.*

Proof. Let X have a dense Lindelöf subspace D, and $\mathcal{U}$ be an open cover of X. Since D is a dense Lindelöf subset of X, then there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $D \subseteq \bigcup \mathcal{V}$. Hence, $St(\bigcup \mathcal{V}, \mathcal{U}) = X$, which shows that X is star-Lindelöf.■

The following two propositions are straightforward.

Proposition 3.1.5. [5].

1. *X is star-compact iff for every open cover $\mathcal{U}$ there is a finite subset $F \subseteq X$ such that for every $x \in X$, $St(\{x\}, \mathcal{U}) \cap F \neq \emptyset$.*
2. *X is weakly star-compact iff for every open cover $\mathcal{U}$ there is a finite subset $F \subseteq X$ such that for every $x \in X$, $\overline{St(\{x\}, \mathcal{U})} \cap F \neq \emptyset$.*

Proposition 3.1.6. [23]. *A space X is neighborhood star-Lindelöf iff for every open cover $\mathcal{U}$ of X there exists a countable subset $C \subseteq X$ such that for every $x \in X$, $\overline{St(\{x\}, \mathcal{U})} \cap C \neq \emptyset$.*

These two properties (i.e. weakly star-compact and neighborhood star-Lindelöf) are between the Hausdorff condition and regularity.

Recall that a space is called Urysohn if every two distinct points have open neighborhoods with disjoint closures.

Theorem 3.1.7. [5]. *A Urysohn weakly star-compact space is countably compact.*

Proof. Let A be an infinite, countable closed discrete subspace of a Urysohn space X. Enumerate $A = \{a_{m,n} : m = 1,2, \ldots; n = 1,2, \ldots, m\}$ and choose open sets $U_{m,n}$ such that $U_{m,n} \cap A = \{a_{m,n}\}$ and $\overline{U_{m,n}} \cap \overline{U_{m,k}} = \emptyset$ whenever $n \neq k$. Then the open cover $\{X \backslash A\} \cup \{U_{m,n} : m = 1,2, \ldots; n = 1,2, \ldots, m\}$ does not fit 2. of proposition 3.1.5.■

So, for Urysohn spaces, the three properties, countable compactness, star-compactness, and weak star-compactness, are equivalent. The previous theorem can be improved using Bonanzinga modification for the Urysohn spaces definition.

Definition 3.1.8. [5]. (Modification of Urysohn spaces, n-Urysohn spaces). Let $n \geq 2$. Distinct points $x_1, \ldots, x_n \in X$ are called *cl-dislinked* if there are neighborhoods $U_1, U_2, \ldots, U_n$ with $x_i \in U_i, \forall\, i \in \{1,2, \ldots, n\}$ such that $\overline{U_1} \cap \ldots \cap \overline{U_n} = \emptyset$. Say that X is *n-Urysohn* if every n distinct points are cl-dislinked.

Clearly, 2-Urysohn is Urysohn, and that, for every $n \geq 2$, n-Urysohn implies (n+1)-Urysohn. The irrational slope topology in [24, Example 75] is 3- Urysohn but not Urysohn.

Theorem 3.1.9. [5].(A Generalization of Theorem 3.1.7) *If X is weakly star-compact space and n-Urysohn for some $n \geq 2$, then X is countably compact.*

Proof. Assume X is not countably compact, let D be an infinite, closed discrete subspace of X. For each $a \in D$, pick a neighborhood V_a such that $V_a \cap D = \{a\}$.

For $m \geq 1$, pick a pairwise disjoint subsets $D_m \subseteq D$ of cardinality $(m.(n-1)+1)$.

For every $m \geq 1$, every subset $A \subseteq D_m$ of cardinality n, and every $a \in A$, select a neighborhood $U_{A,a}$ such that $a \in U_{A,a}$ so that $\bigcap\{\overline{U_{A,a}} : a \in D_m\} = \emptyset$.

For $m \geq 1$, and $a \in D_m$, put $U_a = V_a \cap (\cap \{U_{A,a} : a \in A \subseteq D_m \ and \ |A| = n\})$. Then the family $\{\overline{U_a} : a \in D_m\}$ has the property (*)

(*) *every n-element subfamily has empty intersection*.

It follows from (*) that

for every $F \subseteq X$ *such that* $|F| \leq m$, *not all sets* $\overline{U_a}, a \in D_m$, *meet* F

(indeed, by (*), each point of F can be contained in at most $n-1$ sets $\overline{U_a}, a \in D_m$; since $|F| \leq m$, F intersects at most $(m.(n-1))$ such sets, and we have $|D_m| = m.(n-1)+1$).

Taking together all m, we get that

for every finite $F \subseteq X$, not all sets $\overline{U_a}, a \in \cup \{D_m : m \geq 1\}$, meet F.

Consider this open cover of X : $\mathcal{U} = \{U_a : a \in \cup \{D_m : m \geq 1\}\} \cup \{X \backslash \cup \{D_m : m \geq 1\}\}$. Thus X is not weakly star-compact if we consider the described open cover $\mathcal{U}$.■

[5, Example 3.1] presents an example of a Hausdorff weakly star-compact space which is not countably compact (or, equivalently, not star-compact), indeed the example can considered as a modification of example 75 in [25].

3. 2. Some Properties of Weakly Star-Compact spaces and Neighborhood Star-Lindelöf Spaces

Unlike countable compactness, a closed subspace of a weakly star-compact space need not be weakly star-compact. We can consider this fact as a simple proposition which is similar to the fact that if all closed subspaces of a Tychonoff space are pseudocompact, then the space is countably compact [5].

Proposition 3.2.1. [5]. *The following conditions in a space X are equivalent:*

1. *All closed subspaces of X are weakly star-compact.*
2. *X is countably compact.*

Proof. $(2) \Rightarrow (1)$ obvious. To prove that (1) ⟹ (2) it is enough to notice that an infinite discrete space obviously is not weakly star-compact. ■

Example 3.2.2. (The Isbell-Morwka Space)

Let $X = \mathbb{N} \cup \mathcal{R}$ be the Isbell-Morwka space (see [15]), where $\mathcal{R}$ is a maximal almost disjoint family of subsets of $\mathbb{N}$ with $|\mathcal{R}| = \mathfrak{c}$. Consider the basic open sets on X as follows:

- If $x \in \mathbb{N}$, then the basic open set of x is $\mathcal{O}(x) = \{x\}$.
- If $x \in \mathcal{R}$, so assume $x = N \subseteq \mathbb{N}$, take $\mathcal{O}(x) = \mathcal{O}(N) = \{N\} \cup N \backslash S$. Where S is a finite subset of N. (i.e $\mathcal{O}(N) = \{\{N\}, N \backslash S\}$)

The space X above is Tychonoff pseudocompact, and strongly star-Lindelöf, because $\mathbb{N}$ is a countable dense subset of X. Thus X is neighborhood star-Lindelöf.

Example 3.2.3. [23]. A closed subset of a Tychonoff neighborhood star-Lindelöf space X need not be neighborhood star-Lindelöf.

Proof. Consider the Isbell-Morwka space shown in Example 3.2.2, since $\mathcal{R}$ is a discrete closed subset of cardinality $\mathfrak{c}$ and not neighborhood star-Lindelöf.

Example 3.2.4. There exists a pseudocompact, neighborhood star-Lindelöf Tychonoff space X that is not weakly star-compact.

Proof. Cosider the Isbell-Morwka space in example 3.2.2, X is neighborhood star-*Lindelöf* space. But X is not countably compact, since $\mathcal{R}$ is an uncountable discrete subset of X. Thus X is not weakly star-compact, since countable compactness is equivalent to weakly star-compactness in the class of Tychonoff spaces.■

Theorem 3.2.5. *An open F_σ-subset of a neighborhood star- Lindelöf space is neighborhood star- Lindelöf.*

Proof. Let X be a neighborhood star-Lindelöf space and let $Y = \bigcup\{H_n : n \in \mathbb{N}\}$ be an open F_σ-subset of X, where the set H_n is closed in X for each $n \in \mathbb{N}$. To show that X is nieghborhood star-lindelof. Let $\mathcal{U}$ be an open cover of Y. We are looking to find a countable subset A of Y such that for each open O containing A (i.e. $A \subseteq O$), $St(O,\mathcal{U}) = Y$. For each $n \in \mathbb{N}$, consider the open cover $\mathcal{U}_n$ of X. Such that:

$$\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$$

Since X is neighborhood star-Lindelöf, there exists a countable subset F_n of X such that for each open V containing F_n ($F_n \subseteq V$), $St(V,\mathcal{U}) = X$. For each $n \in \mathbb{N}$, let $M_n = F_n \cap Y$. Then M_n is a countable subset of Y such that for each open O containing M_n, $H_n \subseteq St(O,\mathcal{U})$. If we put $F = \bigcup\{M_n : n \in \mathbb{N}\}$, then F is a countable subset of Y, such that for each open O containing F, $St(O,\mathcal{U}) = Y$. Which shows that Y is a neighborhood star- Lindelöf.■

It is known that a continuous image of a strongly star-lindelof is strongly star-lindelof (see [9]), similarly, we have the following:

Theorem 3.2.6. [5]. *A continuous image of neighborhood star-Lindelöf space is neighborhood star-Lindelöf.*

Proof. Let $f: X \to Y$ be a continuous mapping from a neighborhood star-Lindelöf space X onto a space Y. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U}) = \{f^{-1}(U): U \in \mathcal{U}\}$ is an open cover of X. Since X is a neighborhood star-Lindelöf space, there exists a countable subset A of X such that for every open O containing A, $St(O, f^{-1}(\mathcal{U})) = X$. Then $f(A)$ is a countable subset of Y such that for every open W containing $f(A)$, $St(W, \mathcal{U}) = Y$.

In fact, let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$; let W be an open subset of Y such that $f(A) \subset W$. Hence $f^{-1}(W)$ is an open subset of X such that $A \subseteq f^{-1}(W)$ and then $St(f^{-1}(W), f^{-1}(\mathcal{U})) = X$, that is there exists $U \in \mathcal{U}$ such that $x \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(W) \neq \emptyset$. Then $y = f(x) \in f(f^{-1}(U)) = U$ and $U \cap W \neq \emptyset$. This means $y \in St(W, \mathcal{U})$.■

Similarly, we have the next result for the weakly star-compact spaces.

Theorem 3.2.7. [5]. *A continuous image of a weakly star-compact space is weakly star-compact.*

Proof. Let $f: X \to Y$ be a continuous mapping from a weakly star-compact space X onto a space Y. Let $\mathcal{U}$ be an open cover of Y. Then $f^{-1}(\mathcal{U}) = \{f^{-1}(U): U \in \mathcal{U}\}$ is an open cover of X. Since X is weakly star-compact, there exists a finite subset F of X such that for every open O containing F, $St(O, f^{-1}(\mathcal{U})) = X$. Then $f(F)$ is finite subset of Y such that for every open W containing $f(F)$, $St(W, \mathcal{U}) = Y$.

In fact, let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$; let W be an open subset of Y such that $f(F) \subseteq W$. Hence $f^{-1}(W)$ is an open subset of X such that $F \subseteq f^{-1}(W)$ and then $St(f^{-1}(W), f^{-1}(\mathcal{U})) = X$, that is there exists $U \in \mathcal{U}$ such that $x \in f^{-1}(U)$ and $f^{-1}(U) \cap f^{-1}(W) \neq \emptyset$. Then $y = f(x) \in f(f^{-1}(U)) = U$ and $U \cap W \neq \emptyset$. This means $y \in St(W, \mathcal{U})$.■

Example 3.2.8. [23]. *There exists a neighborhood space X and a compact space Y such that $X \times Y$ is not neighborhood star-Lindelöf.*

Proof. Let X be the Isbell-Morwka space, then $X = \mathbb{N} \cup \mathcal{R}$, where $\mathcal{R}$ is a maximal almost disjoint family of subsets of $\mathbb{N}$ with $|\mathcal{R}| = \mathfrak{c}$. Then X is neighborhood star-Lindelöf space.

Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{y_*\}$ be the one-point compactification of D.

We want to show that $X \times Y$ is not neighborhood star-Lindelöf space. Now since $|\mathcal{R}| = \mathfrak{c}$, we can enumerate $\mathcal{R}$ as $\{r_\alpha : \alpha < \mathfrak{c}\}$. Let

$$U_n = \{n\} \times Y, for\ each\ n \in \mathbb{N},$$

$$V_\alpha = X \times \{d_\alpha\}, for\ each\ \alpha < \mathfrak{c}.$$

And $W_\alpha = (\{r_\alpha\} \cup \omega) \times (Y \backslash \{d_\alpha\}), for\ each\ \alpha < \mathfrak{c}.$

Let

$$\mathcal{U} = \{U_n : n \in \omega\} \cup \{V_\alpha : \alpha < \mathfrak{c}\} \cup \{W_\alpha : \alpha < \mathfrak{c}\}.$$

Then $\mathcal{U}$ is an open cover of $X \times Y$. Observe that $\langle r_\alpha, d_\alpha \rangle \in U \in \mathcal{U}$ iff $U = V_\alpha$. Let us consider the open cover $\mathcal{U}$ of $X \times Y$. It is enough to show that for any countable subset A of $X \times Y$, there exist a point $\langle x, y \rangle \in X \times Y$ such that $\overline{St(\langle x, y \rangle, \mathcal{U})} \cap A = \emptyset$ (by proposition 3.1.6). Let A be any countable subset of $X \times Y$. Then there exists $\alpha < \mathfrak{c}$ such that $A \cap V_\alpha = \emptyset$. Since V_α is the only element of $\mathcal{U}$ containing the point $\langle r_\alpha, d_\alpha \rangle$, then $\overline{St(\langle r_\alpha, d_\alpha \rangle, \mathcal{U})} = V_\alpha$, which shows that $X \times Y$ is not neighborhood star-Lindelöf.■

We can notice that Example 3.2.8 shows that the preimage of a neighborhood star-Lindelöf space under an open perfect map need not be neighborhood star-Lindelöf space. We close this section by the following theorem:

Theorem 3.2.9. [5]. *If X is not countably compact, and Y contains a family of open sets $U_n, n \in \mathbb{N}$, such that $\overline{U_n} \cap \overline{U_m} = \emptyset$ whenever $n \neq m$, then $X \times Y$ is not weakly star-compact.*

Proof. Let $A = \{a_n : n \in \mathbb{N}\}$ be a closed discrete subset of X. For each $n \in \mathbb{N}$ pick an open set $O_n \subseteq X$ so that $O_n \cap A = \{a_n\}$. Also, pick $b_n \in U_n$ and denote $c_n = \langle a_n, b_n \rangle$, and $C = \{c_n : n \in \mathbb{N}\}$. Then the open cover $\mathcal{U} = \{O_n \times U_n : n \in \mathbb{N}\} \cup \{X \times Y \setminus C\}$ *witness that $X \times Y$ is not weakly star-compact.*■

Corollary 3.2.10. [5]. *If $X \times (\omega_1 + 1)$ is weakly star-compact, then X is countably compact.*

Corollary 3.2.11. [5]. *The perfect, open preimage of a weakly star-compact space need not be weakly star-compact.*

Proof. Consider the projection $\pi: X \times Y \to X$ where X is a Hausdorff, weakly star-compact non countably compact space and Y is the *ordinal space* $\omega_1 + 1$.■

Chapter 4

Star-P Spaces

In this chapter, we study the topological property $star - P$ in general, and for $P \in \{finite, countable, compact, Lindelöf\}$ as well known topological properties.

4. 1. Some Definitions and Implications Related To Star-*P* Covering Spaces.

In this section we shall study Alas's approach in some details to present some relations between her definitions and the previous terminologies. To be more specific, the term of star-P was firstly used in [14] by van Mill et al. when he tried to develop the idea of van Douwen and defined the dual class, his idea gives rise to new classes of spaces determined by stars of open covers. Namely, if P is a class (or a property) of spaces then X is star-P (or star determined by P) if, for any open cover $\mathcal{U}$ of the space X, there is a subspace $Y \subset X$ with $St(Y, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap Y \neq \emptyset\} = X$ and $Y \in P$.

It was observed in [13] that star-pseudocompactness is equivalent to pseudocompactness; the term "star-compact", "star-finite" and some others have been used by some authors in a completely context. For example, in [9] the term "n-star-compact" was defined for any natural n; on the other hand, the terms "star-compact" and "star-Lindelöf" are equivalent to "star-finite" and "star-countable", respectively. In [13], the terms star compact and strongly 1-star compact have been used in place of the concept "star-finite". And the term "strongly 1-star$-Lindelöf$" used is equivalent to the property "star-countable".

However, it is well known that being star-finite is equivalent to countable compactness in the class of Hausdorff spaces [11], so studying star properties basically consists of looking at generalization of the class of countably compact spaces. This notations shows that there is no established tradition in naming the above mentioned notations. Later, at 2011 O. Alas and et al. [3] was concern about this class of spaces more extensively than van Mill did especially for

$$P \in \{finite, countable, compact, Lindelöf\}$$

And their relation with other properties.

In [3] the authors investigate, among the other things, the relationship between the star-Lindelöf and star-countable properties, and pseudocompactness. L.P Aiken [1] was investigated star covering properties of Ψ-spaces, and answering some questions listed in [3]. O. Alas et al. in [2], showing that a dense pseudocompact subspace of a Tychonoff cube need not be star-Lindelöf and there exist pseudocompact spaces with a point countable base which are not star-Lindelöf.

We caution to check each author's usage of terminology as it varies from author to author. We introduce the new definitions by the point view of Alas in [3] since their terminology seems simple, direct, and logical. (as we see, van Mill was just used this term by his way to define the dual classes, he didn't focus on this term as O. Alas et al. does). We use it systematically in this chapter. And we are going to use O. Alas et al. concept to rewrite and improve some results, definitions and theorems. We are also trying to investigate some theorems related star-P covering spaces.

Definition 4.1.1. [3]. Let P be a topological property. A space X is said to be $star-P$ if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subseteq X$ with property P such that $X = St(A, \mathcal{U})$. The set A will be called a *star kernel of the cover* $\mathcal{U}$.

One of the motivations to study star-P properties is a folklore fact that every space is star-discrete (and hence star-metrizable); the next proposition illustrate this fact.

Proposition 4.1.2. [2]. Every space is star discrete.

Proof. In fact, if we are given any open cover $\mathcal{U}$ of X, there is a star kernel of $\mathcal{U}$ which is a closed and discrete subspace of X. (i.e. there exists a closed discrete set $D \subset X$ such that $St(D, \mathcal{U}) = X$). To see it, choose inductively a point $x_a \notin St(\{x_\beta : \beta < \alpha\}, \mathcal{U})$. If μ is the first ordinal (which is impossible), then $D = \{x_\alpha : \alpha < \mu\}$ is a closed discrete subset of X and $St(D, \mathcal{U}) = X$. ■

Also, we can see that if a space is not star-countable, then it has an uncountable closed and discrete subspace. Thus we have the following modified result.

Theorem 4.1.3 [4] A T_1-space of countable extent is star-countable.

Proof. If X is not star-countable, then there exists an open cover $\mathcal{U}$ of X such that $St(C, \mathcal{U}) \neq X$ for every countable subset C of X. Thus we can define a sequence of points $x_\alpha, \alpha < \omega_1$ such that $x_\alpha \notin St(\{x_\beta : \beta < \omega_1\}, \mathcal{U})$ for each $\alpha < \omega_1$. Then the set $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable discrete and closed subset of X, which is a contradiction.■

The converse of the above theorem need not be true, the topological space Moore plane Γ contains (which is star-countable) an uncountable discrete closed subspace.

Remark 4.1.4. [3]. *If K is compact and $\mathcal{U}$ is an open cover of $X \times K$ by basic open sets, then for each $x \in X$, there is an open set W_x in X such that $W_x \times K$ is covered by a finite number of elements of $\mathcal{U}$, say*

$$W_x \times K \subseteq \bigcup \{U_k(x) \times V_k(x) : 1 \leq k \leq n_x\}$$

Where $W_x \subseteq \bigcap \{U_k(x) : 1 \leq k \leq n_x\}$.

Definition 4.1.5. [3]. A topological property P is called *compactly productive* if whenever X has P and Y is compact, then $X \times Y$ has P.

For example, being $Lindelöf$ and $\sigma - compact$ are compactly productive properties. Countability is clearly not a compactly productive property.

Theorem 4.1.6 [3]. *If P is a compactly productive property, then so is $star - P$.*

Proof. Suppose that X is $star - P$ and K is compact. Let $\mathcal{U}$ be a basic cover for $X \times K$. Using the notation of Remark 4.1.4, let $\mathcal{W} = \{W_x : x \in X\}$. Since X is $star - P$, there is a subspace $A \subseteq X$ with property P such that $St(A, \mathcal{W}) = X$. Then $A \times K$ has property P and $St(A \times K, \mathcal{U}) = X \times K$.■

Corollary 4.1.7. *The product of a star-Lindelöf (star $-compact$) space and a compact space is star-Lindelöf (star $\sigma - compact$).*

Theorem 4.1.8. [1]. *Suppose X has uncountable closed discrete subspace D whose points can separated by pairwise disjoint open sets, then, X is not star-countable.*

Proof. Choose $D \subseteq X$ as in the hypothesis. For each $x \in D$, let $U_x \subseteq X$ be an open set containing x such that for each $y \in D \backslash \{x\}$, $U_x \cap U_y = \emptyset$. Then $\mathcal{U} = \{U_x : x \in D\} \cup \{X \backslash D\}$ is an open cover for which there is no countable star-kernel of the cover $\mathcal{U}$.■

Theorem 4.1.9. [1]. *Suppose X is locally-countable. Then the following are equivalent:*

1. *X is star-countable.*
2. *X is star-Lindelöf.*

Proof. The equivalence is immediate from the fact that in a locally countable space, every Lindelöf subspace is countable and contained in an open Lindelöf subspace.■

Theorem 4.1.10. [1]. *If X is a Hausdorff first-countable space and $ext(X) > c$, then X is not star-countable.*

Proof. Fix a closed discrete subset $D \subseteq X$ such that $|D| > c$. For each $x \in D$, let $\{U_{x,n} : n < \omega\}$ enumerate a countable base at x such that $U_{x,n} \cap D = \{x\}$. For $n < \omega$, let $\mathcal{U}_n = \{U_{x,n} : x \in D\} \cup \{X \backslash D\}$. Suppose that $Y_n \subseteq X$ is countable such that $St(Y_n, \mathcal{U}_n) = X$. Since $U_{x,n}$ is the only open set in $\mathcal{U}_n$ containing x, then $\mathcal{U}_{x,n} \cap Y_n \neq \emptyset$. Thus $D \subseteq \overline{\bigcup_{n<\omega} Y_n}$, which is contradicting Theorem 4.1.9.■

Next, the Lemma mentioned in [3] without proof.

Theorem 4.1.11. *Both separable and* $Lindelöf$ *spaces are star-countable.*

Proof. Let X be a separable space, and let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X then there exists a countable dense subset $D \subseteq X$. Thus $D \cap U_\alpha \neq \emptyset, \forall \alpha \in \Delta$. Which means that $St(D, \mathcal{U}) = X$.

Let X be a $Lindelöf$ space, and let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X, consider $\acute{\mathcal{U}} = \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}, \ldots\} \subseteq \mathcal{U}$ is an open countable subcover of X such that $n \in \mathbb{N}$, then $\{\alpha_1, \ldots, \alpha_n, \ldots\}, n \in \mathbb{N}$ is a countable subset of X such that $\alpha_n \in U_n, \forall n \in \mathbb{N}$. Hence $St(\{\alpha_1, \ldots, \alpha_n, \ldots\}, \mathcal{U}) = X$, which means that the space X is star-countable.■

Theorem 4.1.12. *Star-countable ⇒ star σ-compact ⇒ star-*$Lindelöf$.

Proof. Let X be a star-countable space, and let $\mathcal{U}$ be an open cover for X. Since X is star-countable, so there exists a countable subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. Since $A = \bigcup_{x_n \in A} \{x_n\}, n \in \mathbb{N}$, so A is $\sigma - compact$star kernel of the cover $\mathcal{U}$.

Let X be a star $\sigma - compact$ space, and let $\mathcal{U}$ be an open cover of X. Since X is star $\sigma - compact$, so there exists a $\sigma - compact$ subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. Since $= \bigcup_{n \in \mathbb{N}} A_n$; A_n is compact $\forall n \in \mathbb{N}$. Hence A is $Lindelöf$ star kernel of the cover $\mathcal{U}$.■

Theorem 4.1.13. *Star-finite ⇒ star-compact ⇒ star-*$Lindelöf$.

Proof. Let X be a star-finite space, and let $\mathcal{U}$ be an open cover of X. Since X is star-finite, there exists a finite subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. But every finite subset is compact. Hence A is compact star kernel of the cover $\mathcal{U}$.

Let X be a star-compact space, and let $\mathcal{U}$ be an open cover of X. Since X is star-compact, there exists a compact subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. But every compact subset is $Lindelöf$. Hence A is $Lindelöf$ star kernel of the cover $\mathcal{U}$.■

Theorem 4.1.14. *Star-finite ⇒ star-countable ⇒ star-$Lindelöf$.*

Proof. Let X be a star-finite space, and let $\mathcal{U}$ be an open cover of X. Since X is star-finite, there exists a finite subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. But every finite subset is countable. Hence A is countable star kernel of the cover $\mathcal{U}$.

Let X be a star-countable space, and let $\mathcal{U}$ be an open cover of X. Since X is star-countable, there exists a countable subspace $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. But every countable subset is $Lindelöf$. Hence A is $Lindelöf$ star kernel of the cover $\mathcal{U}$.■

Theorem 4.1.15. [3]. *A space X is star-countable iff it is star-separable.*

Proof. The necessity is obvious. For the sufficiency, suppose that X is star-separable and $\mathcal{U}$ is an open cover of X. Let Z be a separable subspace of X such that $St(Z, \mathcal{U}) = X$ and let D be a countable dense subspace of Z. If $x \in X$, then there is some $z \in Z$ such that $x \in St(\{z\}, \mathcal{U})$ and thus there is some $U \in \mathcal{U}$ such that $x, z \in U$. Since D is dense in Z, there is some $d \in D \cap U$ and so $x \in St(D, \mathcal{U})$.■

If we summing Theorem 4.1.13 and Theorem 4.1.14, we can conclude the following relations between $star - P$ spaces for the topological property $P \in \{finite, countable, compact, Lindelöf\}$ as in the following diagram.

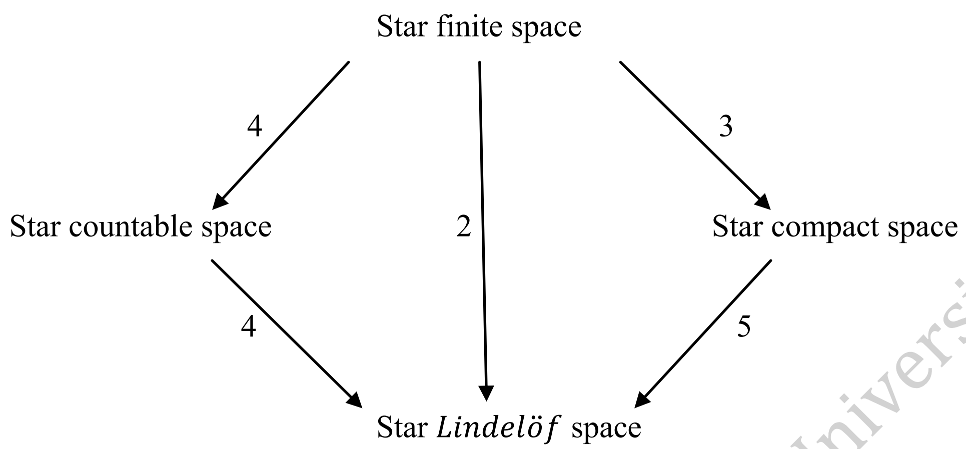

Diagram 4.1

The converses of the above relations do not hold, for instance, the co-finite topology $(\mathbb{R}, \tau_{cof.})$ [Example 3, Appendix 1] is an example that the converses of implications (1) and (3) do not hold. For the converses of the implications (4) and (5) we can consider $(\mathbb{R}, \tau_{coc.})$ [Example 4, Appendix1]. For the converse of implication (2) we can consider any of the above examples [Example 3 or 4, Appendix 1].

For a space which is star countable space, we have no idea if it is star compact space but $(\mathbb{R}, \tau_u)$ is a counter-example for being star countable space but not star compact. The following example shows a space which is star compact space but not star countable.

Example 4.1.16 $(\mathbb{R}^*, \boldsymbol{\tau}^*_{coc.})$

Firstly, look at the spaces that are not star countable and not star compact, try to do 1-point compactification in order to convert the space to star compact space that is not star countable. Try $(\mathbb{R}, \boldsymbol{\tau}_{coc.})$ which is not star compact space, consider $(\mathbb{R}^*, \tau^*_{coc.})$ such that:

- $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$

• $\tau^*_{coc.} = \tau_{coc.} \cup \{ V \subseteq \mathbb{R}^* : \infty \in V \ and \ V^c \equiv$
$(\mathbb{R}^* \backslash V) \ is \ compact \ in \ (\mathbb{R}, \boldsymbol{\tau_{coc.}})\}$

Then :

- $\tau^*_{coc.}$ is a topology on $\mathbb{R}^*$.
- $(\mathbb{R}, \tau_{coc.})$ is a subspace of $(\mathbb{R}^*, \tau^*_{coc.})$.
- $(\mathbb{R}^*, \tau^*_{coc.})$ is a compact space.
- $\mathbb{R}$ is dense in $\mathbb{R}^*$.

Let $\mathcal{U}$ be an open cover for $(\mathbb{R}^*, \tau^*_{coc.})$ such that

$$\mathcal{U} = \{U \subseteq \mathbb{R} : \mathbb{R} \backslash U \ is \ countable\} \cup \{V \subseteq \mathbb{R}^* : \infty \in V \ and \ V^c \ is \ compact \ in \ (\mathbb{R}, \boldsymbol{\tau_{coc.}})\}.$$

To be more specific, consider

$$\mathcal{U} = \{U \subseteq \mathbb{R} : \mathbb{R} \backslash U \ is \ countable\} \cup \{V \subseteq \mathbb{R}^* : \infty \in V \ and \ V^c \ is \ finite\}$$

is the open cover for $(\mathbb{R}^*, \tau^*_{coc.})$.

Now, check that this space is star compact, which is clear since for all open cover $\mathcal{U}$, there exists compact subset $\mathbb{R}^* \subseteq \mathbb{R}^*$ such that $St(\mathbb{R}^*, \mathcal{U}) = \mathbb{R}^*$. To check that this space is not star countable, let $\mathbb{N} \subseteq \mathbb{R}^*$ be a countable subspace of $\mathbb{R}^*$, $0 \in \mathbb{R}, 0 \notin \mathbb{N}$ and consider $\mathcal{V}$ be an open cover for $(\mathbb{R}^*, \tau^*_{coc.})$.

$\mathcal{V} = \{\mathbb{R} \backslash \mathbb{N}, \mathbb{R} \backslash \{0\}\} \cup \{\mathbb{R}^* \backslash \{0\}\}$ then $\mathcal{V}$ is an open cover for $(\mathbb{R}^*, \tau^*_{coc.})$ but $St(\mathbb{N}, \mathcal{V}) = \mathbb{R} \backslash \{0\} \cup \mathbb{R}^* \backslash \{0\} = \mathbb{R}^* \backslash \{0\} \neq \mathbb{R}^*$. So $(\mathbb{R}^*, \tau^*_{coc.})$ not star countable space.

4. 2 Some Topological Properties on Star-*P* Spaces

Theorem 4.2.1. *If every proper subspace of* X *is* $star - P$, *then* X *is* $star - P$.

Proof. Suppose that every proper subspace of X is a $star - P$ space. Let $\mathcal{U}$ be a basic open cover of X. Let $Y = X\setminus\{x_0\}$ be a proper subspace of X. Then from the assumption, Y is $star - P$.

Since $\acute{\mathcal{U}} = \{U \cap Y : U \in \mathcal{U}\}$ is an open cover of Y, then there exists a P subset $D \subseteq Y$ such that $St\big(D, \acute{\mathcal{U}}\big) = Y$.

Let $\mathcal{D} = D \cup \{x_0\}$, then $\mathcal{D}$ is P and $St(\mathcal{D}, \mathcal{U}) = X$. Hence X is $star - P$.■

The next theorem mentioned in [3] without proof.

Theorem 4.2.2. *If* P *is a property preserved under continuous images then the property* $star - P$ *is also preserved under continuous images.*

Proof. Let $f: X \to Y$ be a continuous and onto function and let X be a $star - P$ space. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Therefore $\mathcal{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an open cover of X. Since X is a star-P, so there exists a subspace $A \subseteq X$ with property P such that $St(A, \mathcal{U}) = \cup\{ f^{-1}(V_\alpha) : A \cap f^{-1}(V_\alpha) \neq \emptyset\} = X$. Hence $f(A)$ has property P, since P preserved under continuous images. But we have $A \cap f^{-1}(V_\alpha) \neq \emptyset$, so that $f\big(A \cap f^{-1}(V_\alpha)\big) \neq \emptyset$. And $f\big(A \cap f^{-1}(V_\alpha)\big) = f(A) \cap V_\alpha$. So $f(A) \cap V_\alpha \neq \emptyset$. Therefore, $\cup\{V_\alpha : f(A) \cap V_\alpha \neq \emptyset\} = Y$ which is $St(f(A), \mathcal{V}) = Y$. ■

Thus a continuous image of a star-countable (respectively star$-Lindelöf$, star $\sigma - compact$) space is star-countable (respectively star-$Lindelöf$, star $\sigma - compact$).

In the next proposition, L. Aiken [1] answered a question from the open questions in [3]

Proposition 4.2.3. [1]. *Suppose $f : X \to Y$ is an open perfect map and Y is star-Lindelöf. Then X is star-Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $y \in Y$, choose a finite $\mathcal{U}_y \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}_y \supseteq f^{-1}(y)$ and each $U \in \mathcal{U}_y$ intersects $f^{-1}(y)$. Define an open cover of Y, $\mathcal{V} = \{V_y : y \in Y\}$ where

$$V_y = Y \backslash f[X \backslash (f^{-1}\left(\bigcap_{U \in \mathcal{U}_y} f(U)\right) \cap \bigcup \mathcal{U}_y)].$$

Now, if $q \in V_y$, by the definition of V_y, $f^{-1}(q) \subseteq \bigcup \mathcal{U}_y$ and for each $U \in \mathcal{U}_y$, $U \cap f^{-1}(q) \neq \emptyset$. Since Y is star- Lindelöf, we may choose a Lindelöf subspace $L \subseteq Y$ such that $St(L, \mathcal{V}) = Y$. Let $M = f^{-1}[L]$. Want to show that $St(M, \mathcal{U}) = X$. Choose $x \in X$, $l \in L$ and $y \in Y$ such that $f(x), l \in V_y$. Choose $U \in \mathcal{U}_y$ such that $x \in U$, then $f^{-1}(l)$ intersects U, so $St(M, \mathcal{U}) = X$.

To see that M is Lindelöf, let $\mathcal{U}$ be an open cover of M. For each $l \in L$, choose a finite $\mathcal{U}_l \subseteq \mathcal{U}$ such that $f^{-1}(l) \subseteq \bigcup \mathcal{U}_l$. Let $V_l = Y \backslash f[X \backslash \bigcup \mathcal{U}_l]$ and then define $\mathcal{V} = \{V_l : l \in L\}$. Choose a countable $S \subseteq L$ such that $\{V_l : l \in S\}$ covers L. Then $\bigcup\{\mathcal{U}_l : l \in S\}$ is a countable subcover of M. ■

Theorem 4.2.4. [1]. *The product of a star-countable space and a separable compact space is star-countable.*

Proof. Suppose that X is a star-countable space and K is a separable compact space. Let $\mathcal{U}$ be a basic open cover of $X \times K$. Let $\mathcal{V} = \{V_x : x \in \mathrm{X}\}$(Using Remark 4.1.4).

Now, since X is star-countable space, there is a countable subspace $A \subseteq X$ such that $St(A, \mathcal{V}) = X$. And since K is separable and countable space, there is a countable dense subspace $D \subseteq K$. Therefore, $St(A \times D, \mathcal{U}) = X \times K$. ■

Appendix 1

Now, we are going to check if the well known topologies on $\mathbb{R}$ satisfying $star-P$ property, for P $\in \{finite, countable, compact, lindelöf\}$?

Example 1. *The space* $(\mathbb{R}, \boldsymbol{\tau_{ind.}})$ *is star-finite, star-compact, star-countable and star-Lindelof. Set* $X = \mathbb{R}$. *Let* $\mathcal{U}$ *be an open cover of* X, *there exists a finite subspace* $A \subseteq X$; $A = \{0\}$; *such that* $St(A, \mathcal{U}) = \mathbb{R}$. *So that* $(\mathbb{R}, \tau_{ind.})$ *is star-finite.* $(\mathbb{R}, \tau_{ind.})$ *is star-countable also, since star-finite. Star-compact, since star-finite. And is star-*$Lindelöf$, *since star-compact.*

In general, $(X, \tau_{ind.})$ is star-finite, star-compact, star-countable and star-$Lindelöf$ space. In the next example we show a space that is not star-P for $P \in$ {finite, countable, compact, Lindelöf}.

Example 2. *Consider* $X = (\mathbb{R}, \tau_{dis.})$. *Let* $\mathcal{U}$ *be an open cover of* $(\mathbb{R}, \tau_{dis.})$, *there does not exist a* $Lindelöf$ *subspace* $A \subseteq X$ *such that* $St(A, \mathcal{U}) = \mathbb{R}$. *For a contrary, assume there exists such subspace, namely* $L \subseteq X$, *it is follows that such L should be countable in order to be* $Lindelöf$, *and the star for L could not be all the real numbers, since the basic opens of* $\mathcal{U}$ *are singletons. So* $(\mathbb{R}, \tau_{dis.})$ *is not star-*$Lindelöf$. *It follows that it is not star-compact hence not star-finite. Not star-countable since not star-*$Lindelöf$. *Generally* $(X, \tau_{dis.})$ *has an open cover of singletons, hence it is not star-*$Lindelöf$, *not star-countable, not star-compact and not-star finite.*

Example 3. *Consider* $X = (\mathbb{R}, \tau_{cof.})$. *To check if this space is star-finite, let A be any finite subset of* $\mathbb{R}$ *and let* $y_0 \notin A$. *Consider* $\mathcal{U} = \{ \mathbb{R} \setminus A, \mathbb{R} \setminus \{y_0\}\}$ *be an open cover of* $\mathbb{R}$, *then* $St(A, \mathcal{U}) = \mathbb{R} \setminus \{y_0\} \neq \mathbb{R}$. *So the space is not star-finite. For star-*

countablety, consider any open cover $\mathcal{U}$ *of* $\mathbb{R}$*, then the subset* $\mathbb{N} \subseteq \mathbb{R}$ *is a countable subset of* $\mathbb{R}$ *and* $St(\mathbb{N}, \mathcal{U}) = \bigcup\{\mathbb{R} \setminus V_\alpha : V_\alpha \ is\ finite, (\mathbb{R} \setminus V_\alpha) \cap \mathbb{N} \neq \emptyset, \alpha \in \Delta\} = \mathbb{R}$ *. So the space is star-countable. For star-compactness, let* $\mathcal{U}$ *be an open cover of* $\mathbb{R}$*, since* $(\mathbb{R}, \tau_{cof.})$ *is compact space, so the subset* $\mathbb{R}$ *is compact for every open cover and* $St(\mathbb{R}, \mathcal{U}) = \mathbb{R}$ *. And the space is star-compact. About star-*$Lindel\ddot{o}fness$ *it is obvious (from star-compactness).*

Example 4. *Consider* $X = (\mathbb{R}, \tau_{coc.})$*. To check if this space is star-finite, let* $A \subseteq \mathbb{R}$ *be a finite subset and let* $y_0 \notin A$*. Consider* $\mathcal{U} = \{\mathbb{R} \setminus A, \mathbb{R} \setminus \{y_0\}\}$ *be an open cover of* $\mathbb{R}$*, then* $St(A, \mathcal{U}) = \mathbb{R} \setminus \{y_0\} \neq \mathbb{R}$*. So the space is not star-finite. To check if the space is star-countable, let A be any countable subset of* $\mathbb{R}$ *and let* $y_0 \notin A$*; Consider* $\mathcal{U} = \{\mathbb{R} \setminus A, \mathbb{R} \setminus \{y_0\}\}$ *be an open cover of* $\mathbb{R}$*, then* $St(A, \mathcal{U}) = \mathbb{R} \setminus \{y_0\} \neq \mathbb{R}$*. So the space is not star-countable. To check if the space is star-compact or not, let A be any compact subset of* $\mathbb{R}$ *and let* $y_0 \notin A$*; Consider* $\mathcal{U} = \{\mathbb{R} \setminus A, \mathbb{R} \setminus \{y_0\}\}$ *be an open cover of* $\mathbb{R}$*, then* $St(A, \mathcal{U}) = \mathbb{R} \setminus \{y_0\} \neq \mathbb{R}$ *and the space is not star compact. About star-*$Lindel\ddot{o}f$*, let* $\mathcal{U}$ *be an open cover of* $\mathbb{R}$*, since* $(\mathbb{R}, \tau_{coc.})$ *is* $Lindel\ddot{o}f$ *space, so the subset* $\mathbb{R}$ *is* $Lindel\ddot{o}f$ *for every open cover and* $St(\mathbb{R}, \mathcal{U}) = \mathbb{R}$*. And the space is star-*$Lindel\ddot{o}f$*.*

Example 5. *Consider* $X = (\mathbb{R}, \tau_{l.r.})$*.Check if the space is star finite? For every open cover* $\mathcal{U}$ *of* $\mathbb{R}$*, there exists a finite subspace* $A \subseteq \mathbb{R}$*, take* $A = \{0\}$*, so A is finite and* $St(A, \mathcal{U}) = \mathbb{R}$ *. So the space is star-finite.* $(\mathbb{R}, \tau_{l.r.})$ *is star-countable space since star-finite and thus it's star-*$Lindel\ddot{o}f$ *space. And it's star-compact space since star-finite.*

Similarly $(\mathbb{R}, \tau_{r.r.})$ *is a* $star - P$ *space for the topological property* $P \in \{finite, countable, compact, lindel\ddot{o}f\}$*.*

Example 6. *Consider* $X = (\mathbb{R}, \tau_u)$.To check if the space is star-finite, let $\mathcal{U} = \{ (n-1, n+1) : n \in \mathbb{Z} \}$, so $\mathcal{U}$ is an open cover of $\mathbb{R}$. Let A be any finite subset of $\mathbb{R}$ and let $M = max. A$, then $M + 3 \in \mathbb{R}$ and $M + 3 \notin St(A, \mathcal{U})$. So $(\mathbb{R}, \tau_u)$ is not star-finite. $(\mathbb{R}, \tau_u)$ is star-countable, let $\mathcal{U}$ be any open cover of $(\mathbb{R}, \tau_u)$. Take $A = \mathbb{Q}$, so A is countable and $St(A, \mathcal{U}) = \mathbb{R}$. To show that $St(A, \mathcal{U}) = \mathbb{R}$, suppose there exists $y_0 \in \mathbb{R} \backslash St(A, \mathcal{U})$. Since $\mathcal{U}$ is an open cover of $\mathbb{R}$, so $\exists\, U_0 \in \mathcal{U}$ such that $y_0 \in U_0$. But $U_0 \in \tau_u$, so $\exists\, (a, b) \ni y_0 \in (a, b) \subseteq U_0, for\ some\ a, b \in \mathbb{R}, a < b$. But $(a, b) \cap \mathbb{Q} \neq \mathbb{Q}$, ($\mathbb{Q}$ is dense) which is a contradiction ($U_0 \cap \mathbb{Q} \neq \mathbb{Q}$). $(\mathbb{R}, \tau_u)$ is star $Lindelöf$ space since it is star countable.

$(\mathbb{R}, \tau_u)$ is not star-compact, suppose $\exists$ a compact subset with $St(A, \mathcal{U}) = \mathbb{R}$. Then by Heine-Borel Theorem, A must be bounded and closed. Again let $M = max. A$ and consider $\mathcal{U} = \{(n-1, n+1) : n \in \mathbb{Z}\}$ so $\mathcal{U}$ is an open cover of $\mathbb{R}$. Now $M + 3 \in \mathbb{R}$ and $M + 3 \notin St(A, \mathcal{U})$ so $St(A, \mathcal{U}) \neq \mathbb{R}$ and $(\mathbb{R}, \tau_u)$ is not star compact.

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